Motivation for Combinatorics: A probability model for equally likely outcomes.

Example: There are 4 chips in a box; 1 chip is defective. Randomly draw a chip from the box. What is the probability of selecting the defective chip?

- Common sense → \( P(\text{Draw defective chip}) = \frac{1}{4} \).
- Formally...

\[
\begin{align*}
\Omega &= \{g_1, g_2, g_3, d\} \\
|\Omega| &= 4 \\
A &= \text{draw defective chip} \\
&= \{d\} \\
|A| &= 1 \\
P(A) &= \frac{|A|}{|\Omega|}
\end{align*}
\]

Theorem: If \( \Omega \) is finite with \(|\Omega| = m \) elements, then the assignment that says for \( A \subset \Omega \),

\[
P(A) = \frac{|A|}{|\Omega|} = \frac{\# \text{ outcomes in } A}{\# \text{ outcomes in } \Omega}
\]

produces a valid probability measure with equally likely outcomes.

The thought process for computing probabilities when outcomes are equally likely goes something like this...

Say we want to compute the probability of obtaining a full house when we draw five cards at random from a standard deck. By the theorem,

\[
P(\text{full house}) = \frac{\# \text{ ways to get full house}}{\# \text{ ways to draw five cards}}
\]

is a valid assignment.

When we use the theorem in general, we need to count the number of outcomes in \( A \) and the number of outcomes in \( \Omega \). So, let’s practice counting...
Two Basic Counting Principles:

- Summation Principle: If a complex action can be performed using one of \( k \) alternative methods, \( m_1, \ldots, m_k \), and the methods can be performed in \( n_1, \ldots, n_k \) ways, respectively, then the complex action can be performed in

\[ n_1 + \ldots + n_k \] ways.

- Multiplication Principle: If a complex action can be broken down into a series of \( k \) component actions, performed one after the other, and the first can be performed in \( n_1 \) ways, the second in \( n_2 \) ways, \( \ldots \ ), and the last in \( n_k \) ways, then the complex action can be performed in

\[ n_1 n_2 \cdots n_k \] ways.

**Example**: First toss a coin; then toss a die.
\( \Omega = \{(\text{coin outcome}, \text{die outcome})\} \). What is \( |\Omega| \)?

Sample Selection: Imagine a box with \( n \) objects...
Definitions:

- with replacement: Select a sequence of \( k \) objects. Put the chosen object back in the box after each selection.

- without replacement: Select a sequence of \( k \) objects \( (k \leq n) \). Do not replace an object after it has been selected.

- ordered sample: Keep track of the order in which the objects are selected.

- unordered sample: Order does not matter.

**Example**: A box has the numbers 1, \ldots, 17.

ordered sample : \( (7, 5, 2) \neq (2, 7, 5) \)
unordered sample : \( \{7, 5, 2\} \)
1. Ordered Samples With Replacement

Situation: A box has \( n \) items numbered 1, \ldots, \( n \). Select \( k \) items with replacement. (A number can be drawn twice).

Sample Space:

\[
\Omega = \{(x_1, \ldots, x_k) : x_i \in \{1, \ldots, n\}\}
\]
\[
= \{x_1x_2\ldots x_k : x_i \in \{1, \ldots, n\}\}
\]

What is \(|\Omega|\)?

Break the complex action into a series \( k \) single draws. (\( x_i \) is outcome on draw \( i \)).

* \( n \) possibilities for \( x_1 \)
* \( n \) possibilities for \( x_2 \)
  
  :

* \( n \) possibilities for \( x_k \).

Multiplication principle \( \rightarrow |\Omega| = n^k \).
Example: Octal Numbers
A five-digit octal number is a 5-digit number consisting of the digits 0, \ldots, 7.

(i) How many 5-digit octal numbers are there?

(ii) What is the probability that a randomly chosen 5-digit number is an octal number?

Example: Coin Toss
Toss a coin 23 times.

(i) How many sequences of H's and T's are there?
2. Ordered Samples Without Replacement

Situation: A box has \( n \) items numbered 1, \ldots, \( n \). Select \( k \leq n \) items without replacement. (A number is drawn at most once). Keep track of the sequence of selections.

Sample Space:

\[
\Omega = \{(x_1, \ldots, x_k) : x_i \in \{1, \ldots, n\}, x_i \neq x_j\} = \{x_1x_2\ldots x_k : x_i \in \{1, \ldots, n\}, x_i \neq x_j\}
\]

What is \( |\Omega| \)?

Break the complex action into a series \( k \) single draws. (\( x_i \) is outcome on draw \( i \)).

* \( n \) possibilities for \( x_1 \)
* \( n - 1 \) possibilities for \( x_2 \)
  
  : 
  
  * \( n - (k - 1) \) possibilities for \( x_k \).

Multiplication principle \( \rightarrow |\Omega| = n(n-1)(n-1)\ldots(n-(k-1)) = \frac{n!}{(n-k)!} \).

|\( \Omega \)| has a name...

Defn. Permutation - An ordering of \( k \) distinct objects chosen from \( n \) distinct objects.

Defn. Permutation Number \( (P(n,k)) \) - The number of permutations of \( k \) objects from \( n \).

Thrm.

\[
P(n,k) = \frac{n!}{(n-k)!}
\]
Example: Pizza Toppings
A survey question lists seven pizza toppings and asks you to rank your favorite 3.

(i) How many possible answers to the survey question are there?

(ii) One of the choices is “olives”. If all possible rankings are equally likely, what is the probability that a randomly selected survey has “olives” in the top 3?
3. Unordered Samples Without Replacement

*Situation:* A box has \( n \) items numbered 1, \ldots, \( n \). Select \( k \leq n \) items without replacement. (A number is drawn at most once). Keep track of the *set* of numbers selected in the end. (Order does not matter).

*Sample Space:*

\[
\Omega = \{\{x_1, \ldots, x_k\} : x_i \in \{1, \ldots, n\}, x_i \neq x_j\}
\]

What is \(|\Omega|\)?

\[|\Omega| = \text{ # collections of } k \text{ distinct objects chosen from } n.\]

\(P(n, k)\) can be broken into two steps:

* Select a collection of \( k \) from \( n \).
* Order the \( k \) objects.

Multiplication Principle →

\[P(n, k) = (\text{ # collections of } k \text{ from } n)(\text{ # ways to order the } k) = |\Omega|P(k, k).\]

\[|\Omega| = \frac{P(n, k)}{P(k, k)} = \frac{n!}{(n-k)!k!}\]

|\Omega| has a name...

**Defn. Combination** - A subset of \( n \) distinct objects which has \( k \) distinct objects.

**Thrm.** The number of combinations of \( k \) objects chosen from \( n \) is

\[C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}\]
Example: Powerball (without the powerball)
You pick 5 numbers from \(\{1, \ldots, 49\}\) without replacement. The lottery does the same. The order in which the numbers are picked is irrelevant. You win (something) if you and the lottery pick at least three of the same numbers.

(i) What is the probability that all 5 match?

(ii) What is the probability that you win?

Example (Important for later): Coin Toss
Toss a coin 23 times.

(i) How many ways are there to get 17 heads?
(ii) If the coin is fair, what is the probability of getting 17 heads?
4. Unordered Samples With Replacement

Situation: A box has $n$ items numbered 1, \ldots, $n$. Select $k$ items with replacement. (A number can be drawn more than once). Keep track of the set of numbers – the combination – selected in the end. (Order does not matter).

Sample Space:

$$\Omega = \{ \{x_1, \ldots, x_k\} : x_i \in \{1, \ldots, n\} \}$$

What is $|\Omega|$?

* Since the order does not matter, and objects can be sampled more than once, an outcome essentially consists of counts of how many times the distinct objects are selected.
* We can represent an outcome with a sequence of stars and bars, as follows:
  
  Draw a * for each time “1” is selected. Then draw a separating bar.
  Draw a * for each time a “2” is selected. Then draw a separating bar.
  
  * * | * * | * * * | * * * * *
  
  Draw a * for each time an “n” is selected. Then draw a separating bar.

For example, if $k = 10$ and $n = 4$, the pattern below represents an outcome in which two “1’s”, zero “2’s”, three “3’s”, and five “4’s” are selected.

* * | * * | * * * | * * * * *

* The number of outcomes is the number of ways to choose $k$ spots for the $k$ stars and $n - 1$ spots for the $n - 1$ bars from the $k + (n - 1)$ available positions.

* Aha! Now, we have turned the problem into one of sampling without replacement!

$$|\Omega| = \binom{n + k - 1}{k}$$
Example: Survey Sampling
Sometimes, in surveys a sample is selected from a population with replacement. Suppose a region consists of 1,000 40-acre land segments. A with-replacement sample of 100 segments is selected.

(i) If all segments have an equal probability of being selected on a given draw, what is the probability that no segment is selected twice?

Since all possible samples are equally likely,

\[ P(\text{all different}) = \frac{\# \text{ combinations of 100 unique elements from 1,000}}{\# \text{ with-replacement samples of size 1,000}} \]

\[ = \frac{\binom{1000}{100}}{\binom{1100}{100}} \approx 4.9 \times 10^{-5} \]
## Counting Summary

<table>
<thead>
<tr>
<th>Method</th>
<th>Number of Possible Outcomes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ordered Sample With Replacement</td>
<td>$n^k$</td>
</tr>
<tr>
<td>Ordered Sample Without Replacement</td>
<td>$P(n, k) = \frac{n!}{(n-k)!}$</td>
</tr>
<tr>
<td>Unordered Sample Without Replacement</td>
<td>$C(n, k) = \binom{n}{k} = \frac{n!}{(n-k)!k!}$</td>
</tr>
<tr>
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